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## PROPERTIES OF THE SET OF POSITIVITY FOR THE DENSITY OF A REGULAR WIENER FUNCTIONAL

BY

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**ABSTRACT.** — Let  $f$  be an  $\mathbf{R}^d$ -valued Wiener functional, which is smooth and non-degenerate in the sense of the Malliavin calculus. Let  $\mathbf{p}$  be the density, with respect to the Lebesgue measure on  $\mathbf{R}^d$ , of its law. We are interested in the set  $U = \{\mathbf{p} > 0\}$ . We prove that  $U$  is connected. As a consequence, the intrinsic distance  $d_f$  associated with  $f$  on  $U$  is a true distance (in particular, it is finite). We give in the end an answer to a conjecture of Malliavin about  $d_f$ . © Elsevier, Paris.

**RÉSUMÉ.** — Soit  $f$  une fonctionnelle de Wiener à valeurs dans  $\mathbf{R}^d$ , régulière et non dégénérée au sens du calcul de Malliavin, et soit  $\mathbf{p}$  la densité de sa loi, relativement à la mesure de Lebesgue de  $\mathbf{R}^d$ . On s'intéresse à l'ensemble  $U = \{\mathbf{p} > 0\}$ . On démontre d'abord que  $U$  est connexe. Une conséquence est que la distance intrinsèque  $d_f$  associée à  $f$  sur  $U$  est une vraie distance (en particulier, elle est finie). A la fin, on répond à une conjecture de Malliavin sur  $d_f$ . © Elsevier, Paris.

### 1. Introduction

We consider an abstract Wiener space  $(E, H, \mu)$  and we use the usual notation in Malliavin's calculus (*see*, for example, [18], [6],

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[17]). Let  $f$  denote an  $\mathbf{R}^d$ -valued Wiener functional in  $D^\infty$ . Set  $Jf = [\det(\langle Df_i, Df_j \rangle_H, 1 \leq i, j \leq d)]^{1/2}$ . We suppose that  $f$  is non-degenerate, i.e.,  $(Jf)^{-1}$  belongs to any space  $L^p(E, \mu)$ ,  $p \geq 1$ . It then is well-known that the law of  $f$  on  $\mathbf{R}^d$  has a smooth density  $\mathbf{p}_f$  with respect to the Lebesgue measure on  $\mathbf{R}^d$ . Set  $U_f = \{\mathbf{p}_f > 0\}$ .

Several authors have been interested in this set  $U_f$ , because it is involved in many problems of stochastic analysis (see, for example, [1], [2], [4], [5], [7], [16], etc.). Up to day, two kinds of questions have been considered.

The first one is to find characterizations of the points belonging to  $U_f$ . This is a local problem. A first characterization can be found in [5] when the Wiener functional  $f$  comes from the solution of a stochastic differential equation. Later, in [1], a characterization was obtained in the general case where the Wiener functional  $f$  possesses a skeleton. Recently, in [13], two new characterizations have been obtained, which are applicable to any non-degenerate Wiener functional  $f$  in  $D^\infty$ . The question seems, therefore, well understood.

The second one concerns global properties of  $U_f$ . In FANG [7], the following result was proved:

**PROPOSITION 1.1.** – *Let  $\overline{U}_f$  be the closure of  $U_f$ . Then,  $\overline{U}_f$  is connected. If, in addition,  $f$  is real-valued,  $U_f$  is the interior of  $\overline{U}_f$ .*

However, NUALART [17] gave a counter-example which shows that, when  $f$  takes values in  $\mathbf{R}^2$ , the second property claimed in Proposition 1.1 fails in general. The idea of NUALART is to take a Wiener functional  $f$  such that  $U_f$  is an open square in  $\mathbf{R}^2$ . Then, to compose  $f$  with a diffeomorphism  $\varphi$  on  $U_f$  in such a way that  $U_{\varphi \circ f} = \varphi(U_f)$  does not satisfy the property (see a simpler example in §3 below).

Nualart's counter-example reveals a fact: a diffeomorphism  $\varphi$  can transform a Wiener function for which Fang's result holds, to one for which it does not. A solution was then suggested by MALLIAVIN: Instead of the euclidean distance, consider the distance  $d_f$  on  $U_f$  associated with the image Dirichlet form by  $f$  (see [6] and [3], see also §4 below). The reason is that  $d_f$  is invariant by diffeomorphisms, i.e., for any diffeomorphism  $\varphi$  on  $U_f$ ,  $d_{\varphi \circ f}(\varphi(x), \varphi(y)) = d_f(x, y)$ ,  $\forall x, y \in U_f$  (see §4 below). A conjecture of Malliavin then is:

**CONJECTURE 1.2.** – *For any sequence  $(x_n) \subset U_f$ , the distance  $d_f(x_1, x_n)$  tends to infinity, if  $\mathbf{p}_f(x_n)$  tends to zero.*

This conjecture and related questions are considered in this paper. We have discovered that the essential reason for which Fang's result could not be generalized to higher dimensional Wiener functionals, is that it had not been suitably interpreted. In fact, it is easy to check that, for a Wiener functional  $f$  taking values in  $\mathbf{R}^1$ , Proposition 1.1 is equivalent to:

**THEOREM 1.3.** –  $U_f$  is a connected set.

But, moreover, we can prove (see §3) that Theorem 1.3 holds for any  $\mathbf{R}^d$ -valued non-degenerate Wiener functionals in  $D^\infty$ . One may notice that our proof uses the same argument as FANG did in [7], but the result is stronger. This is because we can prove (see §2) that the set  $f^{-1}[U_f^c]$  is a slim set.

Another pleasant fact is that Theorem 1.3 is also an useful tool to study the distance  $d_f$ . It will be showed in §4 that the connection of  $U_f$  implies that  $d_f$  is indeed a true distance on  $U_f$  and it is topologically equivalent to the euclidean distance on  $U_f$ .

Besides the above general facts on the set  $U_f$ , we also provide a concrete answer to Conjecture 1.2. We shall construct in §5 an example which shows that the conjecture does not hold in general.

## 2. Capacities

We recall some elementary facts about the Gaussian capacities. Let  $r \geq 0, p > 1$ . If  $O$  is an open set of  $E$ , we set

$$c_{r,p}(O) = \inf \{ \|h\|_{r,p}; h \in D^{r,p}, h \geq 1 \text{ } \mu\text{-a.s. on } O \}.$$

For an arbitrary set  $A$ , we set

$$c_{r,p}(A) = \inf \{ c_{r,p}(O); O \text{ is open, } O \supset A \}.$$

A set  $B$  is called an  $(r, p)$ -polar set, if  $c_{r,p}(B) = 0$ . It is called a slim set, if it is  $(r, p)$ -polar for any  $r \geq 0, p > 1$ . A function  $f$  is said to be  $c_{r,p}$ -quasi-continuous, if, for any  $\varepsilon > 0$ , there exists a closed set  $F$  such that  $f$  is continuous on  $F$  and  $c_{r,p}(E - F) < \varepsilon$ . A function  $f$  is said to be quasi-continuous, if it is  $c_{r,p}$ -quasi-continuous for any  $r \geq 0, p > 1$ . It is well-known that, for any  $f \in D^\infty$ , there exists a function  $\tilde{f}$  such that  $f = \tilde{f}$ ,  $\mu$ -a.s., and  $\tilde{f}$  is quasi-continuous.

From now on, we consider an  $\mathbf{R}^d$ -valued non-degenerate function  $f \in D^\infty$ . We suppose that  $f$  is quasi-continuous. Denote simply the set  $U_f$  by  $U$ , the density  $\mathbf{p}_f$  by  $\mathbf{p}$ , and the complement of  $U$  in  $\mathbf{R}^d$  by  $U^c$ . We have the following theorem.

**THEOREM 2.1.** – *The set  $f^{-1}[U^c]$  is slim. More precisely, for any  $r \geq 0$ ,  $p > 0$ ,  $\lim_{\varepsilon \downarrow 0} c_{r,p}(\{f \in \{\mathbf{p} \leq \varepsilon\}\}) = 0$ .*

The proof of the theorem follows from the next lemmas.

**LEMMA 2.2.** – *For any  $m \in N^d$ , for any  $p \geq 1$ , the function  $(1/\mathbf{p}(x))(\partial^{|m|} \mathbf{p}(x)/\partial x^m)$  belongs to  $L^p(\mathbf{R}^d, \mathbf{p} dx)$ .*

*Proof.* – Denote by  $\|\cdot\|_q$  the norm in  $L^q(\mathbf{R}^d, \mathbf{p} dx)$ . According to [18], for any  $1 < q \leq \infty$ , there is a constant  $C$  such that

$$\left| \int \frac{\partial^{|m|} u}{\partial x^m}(x) \mathbf{p}(x) dx \right| \leq C \|u\|_q$$

holds for any  $C^\infty$ -function  $u$  with compact support. The lemma follows if we take for  $q$  the conjugate exponent of  $p$ .  $\square$

**LEMMA 2.3.** – *For any  $m \in N^d$ , for any  $p \geq 1$ , for any  $a > 0$ ,*

$$\lim_{\varepsilon \downarrow 0} \int e^{-(a/\varepsilon)\mathbf{p}(x)} \left| \frac{1}{\varepsilon} \frac{\partial^{|m|} \mathbf{p}(x)}{\partial x^m} \right|^p \mathbf{p}(x) dx = 0.$$

*Proof.* – Set  $g_\varepsilon(x) = e^{-(a/\varepsilon)\mathbf{p}(x)} \varepsilon^{-p} |\partial^{|m|} \mathbf{p}(x)/\partial x^m|^p$ . Notice that, if  $\mathbf{p}(x) > 0$ ,  $e^{-(a/\varepsilon)\mathbf{p}(x)} \varepsilon^{-p} \leq e^{-p} (p/a)^p \mathbf{p}(x)^{-p}$ , for any  $\varepsilon > 0$ . Hence, the family of functions  $g_\varepsilon$  are uniformly overestimated by the function  $e^{-p} (p/a)^p \mathbf{p}(x)^{-p} |\partial^{|m|} \mathbf{p}(x)/\partial x^m|^p$  which is integrable thanks to Lemma 2.2. Since  $\lim_{\varepsilon \downarrow 0} g_\varepsilon(x) = 0$ ,  $\mathbf{p}(x) dx$ -a.s., the lemma follows.  $\square$

**LEMMA 2.4.** –  $\lim_{\varepsilon \downarrow 0} \exp\{-(\mathbf{p} \circ f)/\varepsilon\} = 0$ , in  $D^\infty$ .

*Proof.* – Let  $D$  denote the derivative of the Malliavin calculus. Then, for any integer  $n \geq 0$ ,  $D^n[e^{-\mathbf{p} \circ f/\varepsilon}]$  is a sum of products of  $e^{-\mathbf{p} \circ f/\varepsilon}$ ,  $\frac{1}{\varepsilon} \frac{\partial^{|m|} \mathbf{p}}{\partial x^m} \circ f$  and  $D^k f$ ,  $m, k \leq n$ . Lemma 2.3, together with the Schwarz inequality, implies then the lemma.  $\square$

*Proof of Theorem 2.1.* – It is enough to prove  $\lim_{\varepsilon \downarrow 0} c_{r,p}(\{f \in \{\mathbf{p} \leq \varepsilon\}\}) = 0$ . Notice that the function  $\exp\{-\mathbf{p} \circ f/\varepsilon\}$  is  $\geq e^{-1}$  on the set  $\{f \in \{\mathbf{p} \leq \varepsilon\}\}$ . Since it is quasi-continuous, the wanted limit holds, thanks to Lemma 2.4 (cf. [11]).  $\square$

Notice that in our previous work [13], it was proved that  $x \in U^c$ , if and only if  $f^{-1}(\{x\})$  is slirn. Theorem 2.1 strengthens this result. Let us give another precision on the set  $f^{-1}(\{x\})$  before going to the next section.

**THEOREM 2.5.** – *Suppose  $x, y \in U$ . Then, for any  $p > d$ ,*

$$c_{1,p}[(f^{-1}(\{x\}) + H) \cap f^{-1}(\{y\})] = c_{1,p}[f^{-1}(\{y\})] > 0,$$

where  $f^{-1}(\{x\}) + H = \{\xi + h; \xi \in f^{-1}(\{x\}), h \in H\} \subset E$ . In particular,  $(f^{-1}(\{x\}) + H) \cap f^{-1}(\{y\}) \neq \emptyset$ .

*Proof.* – Since  $c_{1,p}[f^{-1}(\{x\})] > 0$  according to [13], there exists a compact set  $K \subset f^{-1}(\{x\})$  such that  $c_{1,p}[K] > 0$ . Notice that  $K + H$  is a  $K_\sigma$ -set and it is invariant by  $H$ . By [10],  $(f^{-1}(\{x\}) + H)^c \subset (K + H)^c$  is a  $(1, p)$ -polar set. We therefore have

$$c_{1,p}[f^{-1}(\{y\})] = c_{1,p}[(f^{-1}(\{x\}) + H) \cap f^{-1}(\{y\})]. \quad \square$$

### 3. Connection of $U_f$

We begin with a simpler example which shows that Proposition 1.1 does not hold in general. It is enough to take  $f = (e^\xi \cos(\zeta), e^\xi \sin(\zeta))$ , where  $\xi$  and  $\zeta$  are independent standard normal random variables. Then, we check easily that  $f$  is non-degenerate, and

$$\mathbf{p}_f(x, y) = \frac{1}{2\pi} (x^2 + y^2)^{-1} \exp \left\{ -\frac{1}{8} [\log(x^2 + y^2)]^2 \right\} \sum_{\theta = \arg(x + iy)} e^{-\theta^2/2}.$$

The set  $U_f$  is  $\mathbf{R}^2 - \{0\}$ .

Now, let us prove Theorem 1.3.

*Proof of Theorem 1.3.* – Suppose that it is not true. There are then two disjoint non-empty open sets  $U_1$  and  $U_2$  such that  $U = U_1 \cup U_2$ . It is clear that we should have  $\mu(f^{-1}(U_1)) > 0$  and  $\mu(f^{-1}(U_2)) > 0$ . Let  $X$  be the Ornstein-Uhlenbeck process associated with  $(E, H, \mu)$ . It is well-known that  $t \rightarrow f(X_t)$  is continuous almost surely. Moreover, Theorem 2.1,

together with [12], implies that  $P[\exists t \geq 0, X_t \in f^{-1}(U^c)] = 0$ . Now, by ergodicity,

$$\lim_{t \uparrow \infty} P[X_0 \in f^{-1}(U_1), X_t \in f^{-1}(U_2)] \\ = \mu(f^{-1}(U_1)) \mu(f^{-1}(U_2)) > 0.$$

Hence, there would exist an  $\omega$  and  $t > 0$  such that  $f(X_0(\omega)) \in U_1$ ,  $f(X_t(\omega)) \in U_2$ ,  $f(X_s(\omega))$  is continuous on  $[0, t]$ ,  $f(X_s(\omega)) \in U \forall s \in [0, t]$ . This yields a contradiction.  $\square$

#### 4. Intrinsic distance

Let us give the definition of  $d_f$ . Using the notations of [18], we set  $\Gamma_{i,j}(f) = \langle Df_i, Df_j \rangle_H$ ,  $1 \leq i, j \leq d$ , and  $\gamma_{i,j}(f)(x) = (1/\mathbf{p}(x)) \langle \delta_x \circ f, \Gamma_{i,j}(f) \rangle$ ,  $x \in U$ . The functions  $\gamma_{i,j}(f)$  belong to  $C^\infty(U)$ . Recall that  $\gamma_{i,j}(f)$  is a version of  $E_\mu[\Gamma_{i,j}(f)|f=x]$ . Since the matrix  $\Gamma(f) = (\Gamma_{i,j}(f))$  is invertible outside of a slim set (see [13], Lemma 3.1), and since the measure  $\delta_x \circ f$  does not charge the slim sets, the matrix  $\gamma(f)(x) = (\gamma_{i,j}(f)(x))$  is also invertible for any  $x \in U$ . We define, for  $x, y \in U$ ,

$$d_f(x, y) = \sup \{|u(x) - u(y)|; u \in C^1(U), \\ \sum_{1 \leq i, j \leq d} \partial_i u(x) \partial_j u(x) \gamma_{i,j}(f)(x) \leq 1, \forall x \in U\}.$$

LEMMA 4.1. – *Function  $d_f$  is invariant by the diffeomorphisms. More precisely, for any diffeomorphism  $\varphi$  from  $U$  onto  $\varphi(U)$ , such that  $\varphi \circ f$  is in  $D^\infty$  and non-degenerate,  $d_{\varphi \circ f}(\varphi(x), \varphi(y)) = d_f(x, y)$ ,  $\forall x, y \in U$ .*

*Proof.* – Let  $g = \varphi \circ f$ . We have  $Dg_i = \sum_k \partial_k \varphi_i \circ f Df_k$ ,  $\mu$ -a.s. So,  $\Gamma_{i,j}(g) = \sum_{k,k'} \partial_k \varphi_i \circ f \partial_{k'} \varphi_j \circ f \Gamma_{k,k'}(f)$ ,  $\mu$ -a.s. We then have

$$\gamma_{i,j}(g)(\varphi(x)) = E_\mu[\Gamma_{i,j}(g)|g=\varphi(x)] \\ = E_\mu[\sum_{k,k'} \partial_k \varphi_i \circ f \partial_{k'} \varphi_j \circ f \Gamma_{k,k'}(f)|f=x] \\ = \sum_{k,k'} \partial_k \varphi_i(x) \partial_{k'} \varphi_j(x) \gamma_{k,k'}(f)(x), \mathbf{p}(x) dx\text{-a.s.}$$

By continuity, this equality holds for all  $x \in U$ . We therefore have the equivalence between  $\sum_{1 \leq i, j \leq d} \partial_i u \partial_j u \gamma_{i,j}(g) \leq 1$  on  $\varphi(U)$  and

$\sum_{1 \leq i, j \leq d} \partial_i (u \circ \varphi) \partial_j (u \circ \varphi) \gamma_{i,j} (f) \leq 1$  on  $U$ , for any  $u \in C^1(\varphi(U))$ . This is sufficient to conclude the lemma.  $\square$

From now on, we fix a non-degenerate  $D^\infty$ -Wiener functional  $f$ . We shall omit the index  $f$ , and write simply  $d$  for  $d_f$ ,  $\gamma$  for  $\gamma(f)$ .

LEMMA 4.2. – *The function  $d$  is finite on  $U^2$ . More precisely,  $d$  is overestimated by the geodesic distance associated with the Riemannian structure defined by the matrix  $\gamma^{-1} = (\gamma_{i,j})^{-1}$ .*

*Proof.* – Denote the sum  $\sum_{1 \leq i, j \leq d} \partial_i u \partial_j u \gamma_{i,j}$  by  $(\gamma \nabla u, \nabla u)$ . Let  $x, y \in U$ . Because  $U$  is connected, there exists a  $C^1$ -function  $\phi$  from  $[0, 1]$  into  $U$  such that  $\phi(0) = x$  and  $\phi(1) = y$ . Let  $u$  be such that  $(\gamma \nabla u, \nabla u) \leq 1$ . Then,

$$\left| \frac{d}{dt} u \circ \phi(t) \right| = \left| \left( \nabla u(\phi(t)), \frac{d\phi}{dt} \right) \right| \\ \leq \left( (\gamma \nabla u, \nabla u)[\phi(t)] \right)^{1/2} \left( \gamma^{-1}[\phi(t)] \frac{d\phi}{dt}, \frac{d\phi}{dt} \right)^{1/2}.$$

This proves that  $d(x, y) \leq \int_0^1 (\gamma^{-1}[\phi(t)] \frac{d\phi}{dt}, \frac{d\phi}{dt})^{1/2} dt < \infty$ . Taking the infimum in this relation, we prove also the second part of the lemma.  $\square$

Denote by  $|\cdot|$  the euclidean norm of  $\mathbf{R}^d$ .

LEMMA 4.3. – *For any  $x \in U$ , if  $(x_n)_{n \geq 0}$  is a sequence in  $U$  such that  $\lim_{n \rightarrow \infty} d(x, x_n) = 0$ , then  $\lim_{n \rightarrow \infty} |x - x_n| = 0$ .*

*Proof.* – Let  $r$  be a positive real number such that  $B_r(x) = \{y; |y - x| \leq r\}$  is contained in  $U$ . There exists  $u \in C^1(U)$  such that  $u(x) \neq 0$  and  $u = 0$  on  $U - B_r$ . Denote by  $\alpha$  the supremum of  $\{(\gamma \nabla u, \nabla u)(x); x \in U\}$ . Obviously  $\alpha$  is positive and, for any  $y$  belonging to  $U - B_r$ , we have  $d(x, y) \geq \alpha^{-1} |u(x)|$ . The results follows directly.  $\square$

As an immediate consequence of Lemma 4.2 and Lemma 4.3, we obtain

THEOREM 4.4. – *Function  $d$  is a distance which is topologically equivalent to the euclidean distance on  $U$ .*

*Remark.* – We can, instead of the matrix  $\gamma = (\gamma_{i,j})$ , consider directly  $\Gamma = (\Gamma_{i,j})$ . We define then a semi-distance by setting

$$D_f(x, y) = \sup \{|u(x) - u(y)|; u \in C^1(U), \sum_{1 \leq i, j \leq d} \partial_i u \circ f \partial_j u \circ f \Gamma_{i,j}(f) \leq 1\}.$$

We show easily that  $D_f$  is also invariant by diffeomorphisms. However,  $D_f$  may be very degenerate: for some  $f$ ,  $D_f$  is identically zero.

## 5. A counter-example

In this section we shall show by a counter-example, that Conjecture 1.2 does not hold in general. Notice that Conjecture 1.2 holds, if the Wiener space under consideration is one-dimensional. So, the simplest way to build a counter-example would be to consider a two-dimensional Wiener space and to construct an  $\mathbf{R}^1$ -valued Wiener functional on it.

Let us do so. We consider the Gaussian space  $(\mathbf{R}^2, \mu)$ , where  $\mu$  denotes the standard Gaussian measure on  $\mathbf{R}^2$ . We should define a real function  $f$  on  $(\mathbf{R}^2, \mu)$  which belongs to  $D^\infty$ , and is non-degenerate. In the case of a two-dimensional Wiener space, the Malliavin derivative is just the usual derivative (in the distribution sense). So, a real non-degenerate  $D^\infty$ -function  $f$  is simply a  $C^\infty$ -function whose derivatives, as well as  $|\nabla f|^{-1}$ , belong to  $\cap_{p>1} L^p(E, \mu)$ .

As  $f$  will be real-valued, the following formula is available to calculate the distance  $d_f$ :

$$d_f(s, t) = \int_s^t g^{-1}(a) da, \quad \text{for } s, t \in U_f \text{ such that } s < t,$$

where  $g(a) = \sqrt{E[|\nabla f|^2 | f = a]}$ . The function  $f$  that we shall construct will be such that  $U_f = (0, +\infty)$  and  $g(a) \geq C \sqrt{a} (-\log a)^{-1/2}$ , for small  $a > 0$ . Thus, it will satisfy  $\int_0^1 g^{-1}(a) da < \infty$ , which contradicts conjecture 1.2.

The definition of function  $f$  begins with several preliminary functions on  $\mathbf{R}^1$ :



$$v(x) = \exp \left\{ -\frac{1}{x} - \frac{1}{1-x} \right\}, \quad \text{if } 0 < x < 1, = 0 \text{ otherwise};$$

$$\varphi(t) = c^{-1} \int_0^t v(x) dx, \quad \text{if } t \geq 0, = 0 \text{ otherwise},$$

$$\text{where } c = \int v(x) dx;$$

$$\psi(t) = 1 - \varphi(t);$$

$$\tau(t) = m + \int_0^t \varphi(t) dt, \text{ where } m = 1 - \int_0^1 \varphi(t) dt,$$

$$\chi(y) = \exp \{ -\sqrt{2 + \tau(y)^2} \}.$$

These are non negative  $C^\infty$ -functions. Both functions  $\varphi$  and  $\psi$  are monotone, they are constant outside of the interval  $(0, 1)$ . The function  $\tau(t)$  is increasing, it is equal to  $m > 0$  on  $(-\infty, 0]$  and is equal to  $t$  on  $[1, +\infty)$ .

We introduce then a function on  $\mathbf{R}^2$  :  $\eta(x, y) = \exp \{ -\sqrt{\tau(x) + \tau(y)^2} \}$ . It is also a  $C^\infty$ -function. Its partial derivatives, for  $x \geq 1$ , are given by:

$$\partial_y \eta(x, y) = -\eta(x, y) (x + \tau(y)^2)^{-1/2} \tau(y) \tau'(y),$$

$$\partial_x \eta(x, y) = -\frac{1}{2} \eta(x, y) (x + \tau(y)^2)^{-1/2}.$$

Now, our function  $f$  is defined as follows:

$$f(x, y) = \eta(x, y) + (1 + \tau(y)^2) \int_x^2 \psi[\chi(y)^{-4} (t - 2) + 1] dt.$$

It is obviously a  $C^\infty$ -function on  $\mathbf{R}^2$ . Its partial derivatives  $\partial_y f$  and  $\partial_x f$  are given by the following formulas: By a change of variables, we get

$$\begin{aligned} \int_x^2 \psi(\chi^{-4}(y)(t-2)+1) dt \\ = \chi(y)^4 \int \psi(v+1) 1_{] \chi^{-4}(y)(x-2), 0[}(v) dv. \end{aligned}$$

So,  $\partial_y f(x, y)$ , for  $x \geq 1$ ,  $y \geq 1$ , is given by:

$$\begin{aligned} \partial_y f(x, y) &= \partial_y \eta(x, y) + (1 + y^2) 4 \chi(y)^3 \chi'(y) \\ &\quad \int \psi(v+1) 1_{[\chi(y)^{-1}(x-2), 0[}(v) dv \\ &\quad + (1 + y^2) \chi(y)^4 \psi(\chi(y)^{-4}(x-2) + 1) 4 \chi(y)^{-5} \chi'(y)(x-2) \\ &\quad + 2y \chi(y)^4 \int \psi(v+1) 1_{[\chi(y)^{-1}(x-2), 0[}(v) dv \\ &= \partial_y \eta(x, y) + 4 \chi(y)^{-1} \chi'(y) (f(x, y) - \eta(x, y)) \\ &\quad + (1 + y^2) \psi(\chi(y)^{-4}(x-2) + 1) 4 \chi(y)^{-1} \chi'(y)(x-2) \\ &\quad + \frac{2y}{1 + y^2} (f(x, y) - \eta(x, y)). \end{aligned}$$

The derivative  $\partial_x f$  is simpler: for  $x \geq 1$ ,  $y \geq 1$ ,

$$\partial_x f(x, y) = \partial_x \eta(x, y) - (1 + y^2) \psi[\chi(y)^{-4}(x-2) + 1].$$

We have the following proposition.

**PROPOSITION 5.1.** – *The above defined function  $f$  is a non-degenerate  $D^\infty$ -function such that  $U_f = (0, +\infty)$  and  $g(a) \geq C \sqrt{a} (-\log a)^{-1/2}$ , for  $a \in (0, e^{-4})$ , where  $C$  is a strictly positive constant. In particular, it makes invalid Conjecture 1.2.*

*Proof.* – It is clear that  $f$  is a  $C^\infty$ -function. It can also be seen that its derivatives, as well as  $|\nabla f|^{-1}$ , all belong to  $L^p(\mathbf{R}^2, \mu)$ , for any  $p \geq 1$ . Consequently,  $f$  is a non-degenerate  $D^\infty$ -functional. On the other hand, according to Theorem 1 in [13], we see easily that  $U_f = f[\mathbf{R}^2] = (0, +\infty)$ . It remains only to prove the lower bound for the function  $g(a)$ .

The square of the function  $g(a)$  is a conditional expectation which can be calculated by the famous coarea formula:

$$E[|\nabla f|^2 | f = a] = \frac{\int |\nabla f(x, y)| \mathbf{p}(x, y) 1_{\{f=a\}} H(dx dy)}{\int |\nabla f(x, y)|^{-1} \mathbf{p}(x, y) 1_{\{f=a\}} H(dx dy)},$$

where  $\mathbf{p}$  denotes the two dimensional Gaussian density function, and  $H$  denotes the one-dimensional Hausdorff measure. For our specific function  $f$ , the measure  $1_{\{f=a\}} H(dx dy)$  takes a simple form. In fact, the function

$f(x, y)$  being strictly decreasing in  $x$ , the equation  $f(x, y) = a$  defines  $x$  as a nice function of  $y$ . Consequently,

$$1_{\{f=a\}} H(dx dy) = \sqrt{x'(y)^2 + 1} dy,$$

$$\text{and } x'(y) = -\partial_y f(x, y) [\partial_x f(x, y)]^{-1}.$$

It comes from these facts that  $g(a)^2$  may be bounded from below by:

$$\frac{\int |\partial_x f(x(y), y)| \mathbf{p}(x(y), y) dy}{\int |\partial_x f(x(y), y)|^{-1} \mathbf{p}(x(y), y) \sqrt{x'(y)^2 + 1} dy}.$$

To estimate the numerator and the denominator, we divide the set  $\{f = a\}$  into three parts according to the cases:  $x \leq 2 - \chi(y)^4$ ,  $2 - \chi(y)^4 < x \leq 2$  and  $2 < x$ . Let  $I_1, I_2, I_3$  be respectively the projections of each of these parts onto the  $y$ -axis. Let  $a \in (0, e^{-4})$  be fixed.

First of all, we consider a point  $(x, y)$  in the second part, *i.e.*,  $f(x, y) = a$  and  $2 - \chi(y)^4 < x \leq 2$ . Necessarily,  $x > 1$  and, as a consequence of  $a < e^{-4}$ ,  $y > 1$ . We first estimate  $x + y^2$ . We use the following inequalities.

$$(2 - \chi(y)^4 - x)^+ \leq \int_x^2 \psi(\chi(y)^{-4}(t - 2) + 1) dt \leq (2 - x)^+.$$

These inequalities imply that  $\eta(x, y) \geq a - (1 + y^2)(2 - x) \geq a - (1 + y^2)\chi(y)^4$ . So, we get the upper bound of  $x + y^2$ :

$$\begin{aligned} \sqrt{x + y^2} &\leq -\log[a - (1 + y^2)\chi(y)^4] \\ &= -\log a - \log(1 - a^{-1}(1 + y^2)\chi(y)^4) \\ &\leq -\log a + (1 - a^{-1}(1 + y^2)\chi(y)^4)^{-1} a^{-1}(1 + y^2)\chi(y)^4 \\ &\leq -\log a + (1 - a^{-1}\chi(y)^3)^{-1} a^{-1}\chi(y)^3 \\ &\leq -\log a + 2a^2, \end{aligned}$$

because  $\chi(y)^2 < a^2 < \frac{1}{2}$  and  $(1 + y^2)\chi(y) \leq 1$ . For the lower bound of  $x + y^2$ , because  $f(x, y) = a$ , we have  $\eta(x, y) \leq a$ , so,  $\sqrt{x + y^2} \geq (-\log a)$ . These bounds imply that  $I_2$  is contained in  $\{y; [(-\log a)^2 - 2]^{1/2} \leq y \leq (-\log a + 2a^2)\}$ .

We next estimate  $|x'(y)|$ . Notice that since  $y \geq 1$  and  $x \geq 1$ , we may use the formulas for  $\partial_x f$  and  $\partial_y f$ . The following inequalities can be easily obtained:

$$\begin{aligned} |\partial_y \eta(x, y)| &= \eta(x, y) (x + y^2)^{-1/2} y \leq a; \\ |\chi^{-1}(y) \chi'(y)| &= (2 + y^2)^{-1/2} y \leq 1; \\ |x - 2| &\leq \chi^4(y) < a^4; \\ f(x, y) - \eta(x, y) &\leq (1 + y^2)(2 - x) \leq (1 + y^2) \chi^4(y) \leq \chi(y)^3. \end{aligned}$$

By substituting into the formula of  $\partial_y f$ , we obtain  $|\partial_y f(x, y)| \leq a + 4(1 + y^2) \chi^4(y) + 4(1 + y^2) \chi(y)^4 + 2y \chi^4(y) \leq Ca$ . In the same way, we obtain

$$\begin{aligned} |\partial_x f(x, y)| &\geq -\partial_x \eta(x, y) \geq C \eta(x, y) (-\log a + 2a^2)^{-1} \\ &\geq C(a - \chi(y)^3) (-\log a + 2a^2)^{-1} \\ &= Ca(1 - a^{-1} \chi(y)^3) (-\log a + 2a^2)^{-1} \\ &\geq Ca(-\log a + 2a^2)^{-1}, \end{aligned}$$

where  $C$  denotes strictly positive constants which can change from line to line. Consequently,  $|x'(y)| \leq C(-\log a + 2a^2)$ .

We can now write the following inequalities:

$$\begin{aligned} &\int_{I_2} \sqrt{x'(y)^2 + 1} dy \\ &\leq \int 1_{[(-\log a)^2 - 2]^{1/2} \leq y \leq (-\log a + 2a^2)} (|x'(y)| + 1) dy \\ &\leq [C(-\log a + 2a^2) + 1] [4a^2(-\log a) + 4a^4 + 2] \\ &\leq C(-\log a), \text{ and} \\ &\int |\partial_x f(x(y), y)|^{-1} \mathbf{p}(x(y), y) \sqrt{x'(y)^2 + 1} dy \\ &\leq Ca^{-1} (-\log a + 2a^2) (2\pi)^{-1} \exp \left\{ -\frac{1}{2} (-\log a)^2 \right\} \\ &\quad \times \int_{I_2} \sqrt{x'(y)^2 + 1} dy \\ &\leq Ca^{-1} (-\log a + 2a^2) (2\pi)^{-1} \exp \left\{ -\frac{1}{2} (-\log a)^2 \right\} (-\log a) \\ &\leq C_1 a^{-1} (-\log a)^2 \exp \left\{ -\frac{1}{2} (-\log a)^2 \right\}. \end{aligned}$$

After this, we consider a point  $(x, y)$  in the first part, *i.e.*,  $f(x, y) = a$  and  $x \leq 2 - \chi(y)^4$ . We can see again that  $x \geq 1, y \geq 1$ . So, we apply the formulas for  $\partial_x f$  and  $\partial_y f$ . We obtain  $|\partial_x f|(x, y) \geq 1 + y^2$ , and  $|\partial_y f(x, y)| \leq a + 4a + 4(1 + y^2) + a$ . This implies that  $|x'(y)| \leq C$ . As for the projection  $I_2$ , we still have  $x + y^2 \geq (-\log a)^2$ . So,  $y^2 \geq (-\log a)^2 - x \geq (-\log a)^2 - 2$ , *i.e.*,  $I_1 \subset \{y; y \geq [(-\log a)^2 - 2]^{1/2}\}$ . But, since  $I_2$ , as well as  $I_3$  (*see below*), is contained in  $\{y; y \leq (-\log a + 2a^2)\}$ , we have also  $I_1 \supset \{y; y > (-\log a + 2a^2)\}$ .

We can now write:

$$\begin{aligned}
 & \int |\partial_x f(x(y), y)| \mathbf{p}(x(y), y) dy \\
 & \geq \int_{I_1} |\partial_x f(x(y), y)| \mathbf{p}(x(y), y) dy \\
 & \geq e^{-2} \int (1 + y^2) (2\pi)^{-1} \exp \left\{ -\frac{1}{2} y^2 \right\} 1_{y \geq ((-\log a) + 2a^2)} dy \\
 & \geq e^{-2} ((-\log a) + 2a^2) (2\pi)^{-1} \exp \left\{ -\frac{1}{2} ((-\log a) + 2a^2)^2 \right\} \\
 & = e^{-2} (-\log a + 2a^2) (2\pi)^{-1} \\
 & \quad \exp \left\{ -\frac{1}{2} (-\log a)^2 - 2(-\log a)a^2 - 2a^4 \right\} \\
 & \geq C_3 (-\log a) \exp \left\{ -\frac{1}{2} (-\log a)^2 \right\}, \text{ and} \\
 & \int_{I_1} |\partial_x f(x(y), y)|^{-1} \mathbf{p}(x(y), y) \sqrt{x'(y)^2 + 1} dy \\
 & \leq \int (2\pi)^{-1} \exp \left\{ -\frac{1}{2} y^2 \right\} 1_{\{ [(-\log a)^2 - 2]^{1/2} \leq y \}} (|x'(y)| + 1) dy \\
 & \leq C ((-\log a)^2 - 2)^{-1/2} \int y (2\pi)^{-1} \\
 & \quad \exp \left\{ -\frac{1}{2} y^2 \right\} 1_{\{ [(-\log a)^2 - 2]^{1/2} \leq y \}} dy \\
 & = C ((-\log a)^2 - 2)^{-1/2} (2\pi)^{-1} \exp \left\{ -\frac{1}{2} ((-\log a)^2 - 2) \right\} \\
 & \leq C_2 \exp \left\{ -\frac{1}{2} (-\log a)^2 \right\}.
 \end{aligned}$$

Finally, we consider a point  $(x, y)$  in the third part, i.e.,  $f(x, y) = a$  and  $x > 2$ . For this point, we have  $x + \tau(y)^2 = (-\log a)^2$ ,  $\partial_x f(x, y) = \partial_x \eta(x, y) = -\frac{1}{2} a (x + \tau(y)^2)^{-1/2}$ , and  $\partial_y f(x, y) = \partial_y \eta(x, y) = -a (x + \tau(y)^2)^{-1/2} \tau'(y)$ . We get then  $I_3 \subset \{y; y < [(-\log a)^2 - 2]^{1/2}\}$  and  $|x'(y)| \leq 2$ . We can write now:

$$\begin{aligned}
 & \int_{I_3} |\partial_x f(x(y), y)|^{-1} \mathbf{p}(x(y), y) \sqrt{x'(y)^2 + 1} dy \\
 & \leq 6 a^{-1} \int_{I_3} (x(y) + \tau(y)^2)^{1/2} \mathbf{p}(x(y), y) dy \\
 & \leq 6 a^{-1} (-\log a) \int (2\pi)^{-1} \exp\left\{-\frac{1}{2} (-\log a)^2\right\} 1_{\{1 \leq y \leq (-\log a)\}} dy \\
 & \quad + 6 a^{-1} (-\log a) \int (2\pi)^{-1} \exp\left\{-\frac{1}{2} y^2 - \frac{1}{2} (-\log a)^2 + \frac{1}{2}\right\} 1_{y \leq 1} dy \\
 & \leq 6 a^{-1} (-\log a) (2\pi)^{-1} \exp\left\{-\frac{1}{2} (-\log a)^2\right\} (-\log a) \\
 & \quad + 6 a^{-1} (-\log a) (2\pi)^{-1/2} \exp\left\{-\frac{1}{2} (-\log a)^2 + \frac{1}{2}\right\} \\
 & \leq C_4 a^{-1} (-\log a)^2 \exp\left\{-\frac{1}{2} (-\log a)^2\right\}.
 \end{aligned}$$

Substituting these estimates into the inequality for  $g(a)^2$ , we get finally:

$$\begin{aligned}
 g(a)^2 & \geq C_3 (-\log a) \exp\left\{-\frac{1}{2} (-\log a)^2\right\} \\
 & \quad [C_1 a^{-1} (-\log a)^2 + C_2 + C_4 a^{-1} (-\log a)^2]^{-1} \\
 & \quad \exp\left\{\frac{1}{2} (-\log a)^2\right\} \\
 & \geq C_3 [C_4 + C_2 + C_1]^{-1} a (-\log a)^{-1}.
 \end{aligned}$$

The proposition is now proved.  $\square$

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